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UNIT I: PROBABILITY AND RANDOM VARIABLES

PART A

1. From a bag containing 3 red and 2 black balls, two balls are drawn at random. Find the probability that they are of the same color.

P (same colour) = $\dfrac{3C_2 + 2C_2}{5C_2} = \dfrac{3+1}{\frac{5*4}{1*2}} = \dfrac{4}{10} = \dfrac{2}{5}$

2..When A and B are 2 mutually exclusive events such that P (A) = $\dfrac{1}{2}$ and P

(B) = $\dfrac{1}{3}$ find $P(A \cup B)$ and $P(A \cap B)$.

When A and B are mutually exclusive $P(A \cup B)$ = P (A) + P (B) = $\dfrac{1}{2} + \dfrac{1}{3} = \dfrac{5}{6}$

$P(A \cap B) = 0$

3. Conditional probability:

The conditional probability of an event B, assuming that event A has happened is denoted by P (B/A) defined as P(B/A) = n(A∩B) / n(A) = P(A∩B) / P(A), Provided P(A) ≠ 0

4. There are 3 unbiased coins and 1 biased coin with head on both sides. A coin is chosen at random and tossed 4 times. If head occurs all the 4 times what is the probability that the biased coin has been chosen?

Let A $_1$ be the event of choosing a coin which is biased and A $_2$ be the event of choosing a unbiased coin. P(A$_1$) = ¼ P (A$_2$) = ¾ . Let B be the event of getting 4 heads when the randomly chosen coin 4 times. P(B/A$_1$) = 1   P

(B/A$_2$) = 4C$_4$ $(1/2)^4$ $(1/2)^0$ = 1/16
P(B) = P(A$_1$) P(B/A$_1$) + P(A$_2$) P(B/A$_2$) = ¼ .1 + ¾ . 1/16 = 0.3125

5. Given the probability density function of a continuous random variable X as follows f(x) = 6x (1-x) 0<x<1 . Find cumulative density function.

CDF F(x) = $\int\limits_0^x$ f(x) dx = $\int\limits_0^x$ 6x (1-x) dx = 6(x^2 /2 − x^3 /3)$\Big|_0^x$ = 3x^2 − 2 x^3 ,0 < x < 1

6. A continuous random variable X can assume any value between x = 2 and x = 5 has a density function given by f(x) = k(x + 1). Find P(X < 4)

$$\int_2^5 f(x)\, dx = 1 \Rightarrow \int_2^5 k(x+1)\, dx = k\left(x^2/2 + x\right)\Big|_2^5 \Rightarrow k(27/2) = 1 \Rightarrow k = 2/27.$$

$$P(X < 4) = \int_2^4 f(x)\, dx = \int_2^4 k(x+1)\, dx = 2/27\int_2^4 (x+1)\, dx = 2/27\,(x^2/2 + x)\Big|_2^4 = 16/27$$

7. Four persons are chosen at random from a group containing 3 men , 2 women and 4 children. Show that the chance that exactly two of them will be children is 10/21

   Let A be the event of selecting 2 children 1 woman and 1 man.

   $n(A) = 4\,C_2 \cdot 2\,C_1\, 3\,C_1$

 Let B be the event of selecting 2 children 2 men

   $n(B) = 4\,C_2 \cdot 3\,C_2$

 Let C be the event of selecting 2 children 2 women

   $n(C) = 4\,C_2 \cdot 2\,C_2$

P(Exactly 2 children) = P $(A\cup B\cup C)$ = P(A) + P(B) + P(C) = 10 / 21

8. If moment generating function M $_X$ (t) = 1/3 e^t + 4/15 e^{3t} + 2/15 e^{4t} + 4/15 e^{5t} . Find the probability mass function of X.

X	1	2	3	4	5
P(X = x)	1/3	0	4/15	2/15	4/15

9. Suppose M$_X$(t) = (0.4 e^t + 0.6)^8 . Find the MGF of Y = 3X + 2

   M $_Y$ (t) = e ^{2t} M $_X$ (3t) = e ^{2t} (0.4 e ^{3t} + 0.6)^8

10. Define MGF. Why it is called so?

   MFG is defined as M $_X$ (t) = E (e ^{tx}) , which generates the moments of the random variable X, so it is called as Moment Generating function.

11. For a binomial distribution mean is 6 and standard deviation is √2.

   Find the first two terms of the distribution.

   <u>Soln:</u>

   For a binomial distribution,
   Mean= np = 6.
   S.D. = √npq = √2,

npq=2

6 X q =2, q= 1/3. P=
1-q = 1- 1/3 = 2/3.
n = 9.

$P(X=x) = nC_x \, p^x \, q^{n-x}$, x = 0, 1, 2, ...,n

$P(X=0) = 9C_0 \, (2/3)^0 \, (1/3)^9 = 0.0005$

$P(X=1) = 9C_1 \, (2/3) \, (1/3)^8 = 0.0009$.

12. **Find p for a binomial variat, if n=6 and 9 P(X- 4) = P(X =2)**

   Soln:

   $P(X=x) = 6C_x \, p^x \, q^{6-x}$, 9P(X = 4) = P(X = 2),

   $9 \times 6C_4 \, p^4 \, q^2 = 6C_2 \, p^2 \, q^4$

   $9 \, p^2 = q^2$,

   $9 \, p^2 \, (1-p)^2$, Since q= 1- p.

   $9p^2 = 1 + p^2 - 2p$

   $8p^2 + 2p -1 =0$,

   [p+ (1/2)] [p − (1/4)] =0, P = ¼.

13. **Determine the distribution whose M.G.F is $M_X(t) = e^{3(e^t - 1)}$. Also**

   Find P(X=1).

   W.K.T. M.G.F of a poisson distribution
   $M_X(t) = e^{\lambda(e^t - 1)}$

   Given $M_X(t) = e^{3(e^t - 1)}$
   $\lambda = 3$
   $P(X= x) = e^{-\lambda} \, \lambda^x / x!$, x =0, 1, 2, ...

   $P(X= x) = e^{-3} \, 3^x / x!$, x =0, 1, 2, ...

$P(X= 1) = e^{-3} 3 / 1!, x =0, 1, 2, ...$

$\qquad = 3e^{-3}$

14. **Find the M.G.F of geometric distribution.**

$M_X (t) = E (e^{tx})$

$\qquad = \sum\limits_{x= 0}^{\infty} e^{tx}. P(X=x)$

$\qquad = \sum\limits_{x= 0}^{\infty} e^{tx}. p\, q^x$

$\qquad = p \sum\limits_{x= 0}^{\infty} (e^t q)^x.$

$\qquad = p [1 - q\, e^t]^{-1}$

$\qquad = p / (1- q\, e^t)$

15. **Identify the distribution with MGF $M_X(t) = (5 - 4\, e^t)^{-1}$**

$M_X (t) = \{ (1/5) / [1- (4/5)e^t] \}$

WKT, MGF of Geometric distribution, M_X

$(t) = p / (1- q\, e^t)$

∴ The Given MGF is the MGF of geometric distribution with parameter.

P= 1 / 5, q = 4/5.

$P[X= x] = (1/5) (4/5)^{x-1}$, x= 1, 2, …….

16. **Determine the distribution whose MGF is $M_X (t) = (1/3)\, e^t\, [e^{-t} - (2/3)]^{-1}$**
   Soln:

The $M_X (t)$ of negative binomial distribution with parameters p and r is

$M_X (t) = p^r [1 - q\, e^t]^{-r}$,

Given $M_X (t) = (1/3) [e^t\, e^{-t} - (2/3)\, e^t]^{-1}$,

$\qquad = (1/3) [1 - (2/3)\, e^t]^{-1}$,

This is the M.G.F of Negative Binomial Distribution with parameters r = 1 and p = 1/3.

17. **Find the MGF of a uniform distribution in (a, b)**

$$M_X (t) = [1 / (b-a)] = \int_{a}^{b} e^{tx} x. \, b$$

$$= [1 / (b-a)] \{e^{tx} / t\} \quad = [e^{bt} - e^{at}] / (b-a)t$$

18. Find the MGF of a R.V. which is uniformly distributed over (-1, 2)

Soln:

$$M_X (t) = [1 / 3] = \int_{-1}^{2} e^{tx} \, dx. = [(e^{2t} - e^{-t}) / 3t], \, t \neq 0$$

$$M_X (t) = [1 / 3] = \int_{-1}^{2} dx. = 1, \text{ for } t \neq 0$$

19. If X has uniform distribution in (-3, 3) find P[|X - 2| < 2]

P.d.f is $f(x) = 1/6$, $-3 < X < 3$

$P[|X - 2| < 2] = P[0 < X < 4]$

$$=(1 / 6) \int_{}^{3} dx = 1 / 2.$$

PART B QUESTIONS

1. A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(X)	0	K	2k	2k	3k	k2	2k2	7k2+k

Find (i) The value of k, (ii) P[1.5 < X < 4.5 / X >2] and (iii) The smallest value of λ for which P(X ≤ λ) < (1/2).

2. A bag contains 5 balls and its not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that the balls in the bag all are white.

3. Let the random variable X have the PDF $f(x) = \dfrac{1}{2} e^{-\frac{x}{2}}$, x >0 Find the moment generating function, mean and variance.

4. A die is tossed until 6 appear. What is the probability that it must tossed more than 4 times.

5. A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs. 10 for each white ball and Rs 5 for each black ball. Find his expectation.

6. In a certain binary communication channel, owing to noise, the probability that a transmitted zero is received as zero is 0.95 and the probability that a transmitted one is received as one is 0.9. If the probability that a zero is transmitted is 0.4, find the probability that (i) a one is received (ii) a one

was transmitted given that one was received

7. Find the MGF and r^{th} moment for the distribution whose PDF is $f(x) = k\, e^{-x}$, $x > 0$. Find also standard deviation.

8. The first bag contains 3 white balls, 2 red balls and 4 black balls. Second bag contains 2 white, 3 red and 5 black balls and third bag contains 3 white, 4 red and 2 black balls. One bag is chosen at random and from it 3 balls are drawn. Out of 3 balls, 2 balls are white and 1 is red. What are the probabilities that they were taken from first bag, second bag and third bag.

9. A random variable X has the PDF $f(x) = 2x$, $0 < x < 1$ find (i) P ($X < \frac{1}{2}$) (ii) P ($\frac{1}{4} < X < \frac{1}{2}$) (iii) P ($X > \frac{3}{4} / X > \frac{1}{2}$)

10. If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(1) Find a   (2) Find the cdf of X

11. If the the moments of a random variable X are defined by E (X^r) = 0.6, r = 1,2,.. Show that P (X =0) = 0.4 P (X = 1) = 0.6, P (X $\ge$ 2) = 0.

12. In a continuous distribution, the probability density is given by $f(x) = kx$ (2 − x) $0 < x < 2$. Find k, mean , varilance and the distribution function.

13. The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2 & \frac{1}{2} \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

Find the pdf of X and evaluate P (|X| $\le$ 1) using both the pdf and cdf

14. Find the moment generating function of the geometric random variable with the pdf $f(x) = p\, q^{x-1}$, $x = 1,2,3..$ and hence find its mean and variance.

15. A box contains 5 red and 4 white balls. A ball from the box is taken our at random and kept outside. If once again a ball is drawn from the box, what is the probability that the drawn ball is red?

16. A discrete random variable X has moment generating function

$$M_X(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5$$ Find E(x), Var(X) and P (X=2)

17. The pdf of the samples of the amplitude of speech wave foem is found to decay exponentially at rate α, so the following pdf is proposed $f(x) = Ce^{-\alpha|x|}$, $-\infty < X < \infty$. Find C, E(x)

18. Find the MGF of a binomial distribution and hence find the mean and variance.

. Find the recurrence relation of central moments for a binomial distribution.

19. The number of monthly breakdowns of a computer is a RV having a poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (a) without a breakdown, (b) Wish only one breakdown, (c) Wish at least one break down.

20. Find MGF and hence find mean and variance form of binomial distribution.

21. State and prove additive property of poisson random variable.

22. If X and Y are two independent poisson random variable, then show that probability distribution of X given X+Y follows binomial distribution.

23. Find MGF and hence find mean and variance of a geometric distribution.

24. State and prove memory less property of a Geometric Distribution.

25. Find the mean and variance of a uniform distribution.

UNIT- II TWO DIMENSIONAL RANDOM VARIABLES

PATR-A

1.State the basic properties of joint distribution of (X,Y) where x and Y are random variables.

$F(x,y) = P [X \le x, Y \le y]$. $F(-\infty,\infty) = 1$; $F(-\infty,y) =$ Marginal distribution of Y and $F(x,\infty) =$ Marginal distribution of X

2. The joint p.d.f. f(X,Y) is $f(x,y)= 6e^{-2x-3y}$, $x \ge o$, $y \ge 0$, find the conditional density of Y given X.

Soln.

Marginal of X is $f(x) = \int_0^\infty 6e^{-2x} e^{-3y}$ dy,

$= 2e^{-2x}$, x>0

Conditiona of Y given x :

$f(y/x) = f(x/y) / f(x)$

$= (6e^{-2x} e^{-3y})/ 2e^{-2x}$

$= 3 e^{-3x}, y \ge 0$.

3. State the basic properties of joint distribution of (x,y) when X and Y are random variables.

Soln.

$F(x,y) = P(X \le x, Y \le y)$

$F(\infty \; \infty) = 1$

$F(\infty, y) =$ Marginal distribution of Y

$F(x, \infty) =$ Marginal distribution of X.

$F(-\infty, y) = 0$, $F(x, -\infty)=0$.

4.If X has an exponential distribution with parameter α find the pdf of $Y = \log X$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = e^y \alpha e^{-\alpha e^{-y}}, -\infty < y < \infty$$

5.The joint probability mass function of (X,Y) is given by P(x,y) = k (2x + 3y) x = 0,1,2 y = 1,2,3. Find the marginal probability distribution of X

X \ Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

$\sum \sum P(x,y) = 1 \Rightarrow 72 k = 1 \Rightarrow k = 1/ 72$

Marginal distribution of X :

X	0	1	2
P(X = x)	18/72	24/72	30/72

6. If f(x,y) = k (1 − x − y), 0 < x,y < ½ . Find K.

½ ½   ½   1/2

$K \int \int (1 - x - y) \, dx \, dy = 1 \Rightarrow K \int ((1 - y)/2 - 1/8) \, dy = 1 \Rightarrow K (3y/8 - y^2/4) \Rightarrow$
k/8=1
  0 0   0   0

   $\Rightarrow K = 8$

7. If X and Y are independent random variables with variances 2 and 3. Find Var(3X + 4Y)

$Var(3X + 4Y) = 3^2 \, Var(x) + 4^2 \, Var(y) = 9 * 2 + 16 * 3 = 66$

8. If X and Y are independent random variables, find covariance between X + Y and X − Y.

$Cov (X + Y, X - Y) = E [(X + Y) - E (X + Y)] [(X - Y) - E (X - Y)]$
$= E [(X - E(X) + Y - E (Y)) (X - E (Y) - (Y - E (Y))]$
$= E [(X - E(X))^2 - (Y - E (Y))^2$
$= Var \, X - Var \, Y$

9. If X and Y have joint probability density function f(x,y) = x + y 0 < x,y < 1, Check whether X and Y are independent

                          1   1
Marginal density function of X is $f_X(x) = \int f(x,y) \, dy = \int (x + y) \, dy = x + \frac{1}{2}$, 0 < x < 1
                          0   0
                          1   1
Marginal density function of Y is $f_Y(y) = \int f(x,y) \, dx = \int (x + y) \, dy = y + \frac{1}{2}$, 0 < y < 1
                          0   0

  $f_X(x) . f_Y(y) = (x + \frac{1}{2}) . (y + \frac{1}{2}) \neq$
f(x,y) X and Y are not independent.

10. The joint pdf of two random variables X and Y is given by f(x,y) = 1/8 x (x-y), 0 < x < 2, -x < y < x . Find f (y/x).

                          x
        $f_X(x) = \int 1/8 \, x \, (x - y) \, dy = x^3/4$, 0 < x < 2
                          -x

        $f(y/x) = f (x,y) / f_X(x) = 1/8 \, x \, (x-y) / x^3/4 = (x-y) / 2x^2$, -x < y < x

11. The joint pdf of (X,Y) is f(x,y) = $6e^{-2x-3y}$, x ≥ 0, y ≥ 0. find the conditional density of Y given X.

Soln.

$$\text{Marginal of X is } f(x) = \int_0^\infty 6e^{-2x-3y}dy$$

$$= 2e^{-2x}, \; x > 0.$$

Conditional of Y given X:

$$f(y/x) = f(x,y) / f(x)$$

$$= 6e^{-2x}e^{-3y} / xe^{-2x}.$$

$$= 3\,e^{-3y}, \; y \ge 0.$$

12. State the basic properties of joint distribution of (X,Y) when X and Y are random variable.

Soln.
$F(x,y) = P(X ≤ x, Y ≤ y)$
$F(\infty, \infty) = 1$.
$F(\infty, y) = $ Marginal distribution of
Y. $F(x,\infty) = $ Marginal distribution
of X. $F(-\infty, y) = 0$, $F(x, -\infty) = 0$.

13. If 2 random variables have the joint density f(x₁, x₂) = x₁x, 0 < x₁ < 1, 0 < x₂ < 2. Find the probability that both random variables will take on values less than 1.

Soln.

$$P[\,x_1 \le 1 , x_2 \le 1] = \int_0^1 \int_0^1 x_1 x_2 \; dx_2 \; dx_1$$

$$= \int x_1 (\,x_2 \,/ 2) \; dx_1 = 1|4$$

14. If X and Y are independent random variables find covariance between X+Y and X – Y.

Soln.

Cor(X+Y, X-Y) = E [{(X+Y)-E(X+Y)} {(X-Y) – E(X-Y)]

= E[{ X- E(X) +Y – E(Y)} { (X-E(Y)) – (Y-E(Y))}]

= E[{X-E(X)}^{2} – {Y – E(Y) }^{2}

= Var X – Var Y.

15. If X and Y are independent random variable with variances 2 and 3, find Var (3X +4Y).

Soln.

Var(3X+ 4Y) = 3^2 Var(X) + 4^2 Var(Y)

= 9 X 2 + 16 X 3

= 66.

16. If the joint pdf of (X,Y) is given by f(x,y) = e $^{-(x+y)}$, x ≥ 0, y ≥ 0 Find E (XY)

$$E\ (XY) = \int_0^\infty \int_0^\infty xye^{-(x+y)}dxdy = \int_0^\infty xe^{-x}dx\int_0^\infty ye^{-y}dy = 1$$

17. The regression lines between two ransom variables X and Y is given by 3x + 4y =10 and 3x + 4y = 12. Find the correlation between two regression lines.

3x + 4y =10 $\Rightarrow$ b$_{yx}$ = $-\dfrac{3}{4}$. 3x + 4y = 12 $\Rightarrow$ b$_{xy}$ = $-\dfrac{1}{3}$

r^2 = $-\dfrac{3}{4} \cdot -\dfrac{1}{3}$ = $\dfrac{1}{4}$ $\Rightarrow$ r = $\dfrac{1}{2}$

18. **Distinguish between correlation and regression**

By correlation we mean the casual relationship between two or more variables. By regression we mean the average relationship between two or more variables.

19. **Why there are two regression lines?**

Regression lines express the linear relationship between two variable X and Y. Since any of them is taken as independent variable, we have two regression lines.

20. **State central limit theorem?**

If X$_1$, X$_2$ … X$_n$ be a sequence of independent identically distributed R.Vs with E (X$_i$) = μ and var (X$_i$) = σ^2 for o = 1,2… and if S$_n$ = X$_1$ + X$_2$ +… X$_n$, then under certain general conditions, S$_n$ follows a normal distribution with mean nμ and variance nσ^2 as n tends to ∞

21.If Y$_1$ and Y$_2$ are two independent random variables, then covariance (Y$_1$, Y$_2$) = 0. IS the converse of the above statement true? Justify your answer.

The converse not true. Consider X ~ N (0,1) and Y = X^2 since X ~ N (0,1). E (X) = 0; E (X^3) = E (XY) = 0 since all odd moments vanish. Cov (x, y) = 0 but X and Y are not independent.

22. Find the value of k, if f(x,y) = k (1- x) (1- y) for 0 < x,y < 1 is to be a joint density function.

We know $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1 \Rightarrow \int_{0}^{1}\int_{0}^{1} k(1-x)(1-y)dxdy = 1$

$k\int_{0}^{1}\left(x - \frac{x^2}{2} - xy + \frac{x^2 y}{2}\right)_{0}^{1} dy = 1 \Rightarrow k\int_{0}^{1}\left(\frac{1}{2} - y + \frac{y}{2}\right)dy = 1 \Rightarrow k\left(\frac{y}{2} - \frac{y^2}{2} + \frac{y^2}{4}\right)_{0}^{1} = 1$

$k\left(\frac{1}{4}\right) = 1 \Rightarrow k = 4$

Part- B

1. If f (x, y) = x+y , 0< x <1, 0< y < 1
   0 , Otherwise
   Compute the correlation cp- efficient between X and Y.

2. The joint p.d.f of a two dimensional randm variable (X, Y) is given by f(x, y) = (8 /9) xy, $1 \le x \le y \le 2$ find the marginal density unctions of X and Y. Find also the conditional density function of Y / X =x, and X / Y = y.

3. The joint probability density function of X and Y is given by f(x, y) = (x + y) /3, $0 \le x \le 1$ & 0<y <2 obtain the egression of Y on X and of X on Y.

4. If the joint p.d.f. of two random variable is given by $f(x_1, x_2) = 6\ e^{-2x1 - 3\ x2}$, x1 > 0, x2 >0. Find the probability that the first random variable will take on a value between 1 and 2 and the second random variable will take on avalue between 2 and 3. \also find the probability that the first random variable will take on a value less than 2 and the second random variable will take on a value greater than 2.

5. If two random variable have hoing p.d.f. $f(x_1, x_2) = (2/ 3) (x_1 + 2x_2)$ 0< x1 <1, 0< x2 < 1

6. Find the value of k, if f(x,y) = k xy for 0 < x,y < 1 is to be a joint density function. Find P(X + Y < 1) . Are X and Y independent.

7. If two random variable has joint p.d.f. $f(x, y) = (6/5) (x + y^2)$, 0 < x < 1 , 0< y <1.Find P(0.2 < X < 0.5) and P(0.4 < Y < 0.6)

8. Two random variable X and Y have p.d.f $f(x, y) = x^2 + (xy / 3)$, $0 \le x \le 1$, $0 \le y \le 2$. Prove that X and Y are not independent. Find the conditional density function

9. X and Y are 2 random variable joint p.d.f. $f(x, y) = 4xy\ e^{-\left(x^2+y^2\right)}$, x ,y ≥ 0, find the p. d. f. of $\sqrt{x^2+y^2}$.

10. Two random variable X and Y have joint f(x y) 2 – x –y, 0< x <1, 0< y < 1.
Find the Marginal probability density function of X and Y. Also find the conditional
density unction and covariance between X and Y.

11. Let X and Y be two random variables each taking three values –1, 0
    and 1 and having the joint p.d.f.

Y	X		
	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.1

Prove that X and Y have different
expections. Also Prove that X and Y are
uncorrelated and find Var X and Var Y

12. 20 dice are thrown. Find the approximate probability tat the sum obtained is
between 65 and 75 using central limit theorem.

13. Examine whether the variables X and Y are independent whose joint density
is $f(x ,y) = x\, e^{-xy - x}$, 0< x , y < ∞
.
14. Let X and Y be independent standard normal random variables. Find the pdf
of z =X / Y.

15. Let X and Y be independent uniform random variables over (0,1) . Find the
PDF of Z = X + Y

UNIT-III CLASSIFICATION OF RANDOM PROCESS

1. **Define a Random process and give an example of a random process.**
   A random process is a collection of random variables $\{X(s,t)\}$ that are function of time $s \in S$ and $t \in T$.

   Example:

   $$x(t) = A\cos(\omega t + \theta)$$

   Where θ = uniform distributed in $(0, 2\pi)$

   A, ω = constants.

2. **State the four types of a stochastical processes.**
   Discrete time, discrete state

   Discrete time, continuous state

   Continuous time, discrete state

   Continuous time, continuous state

3. **If $\{X(s,t)\}$ is a random process, what is tha nature of $X(s,t)$ when (i) S is fixed (ii) t is fixed.**
   (i)   When S is fixed, $X(s,t)$ is a time function.
   (ii)   When t is fixed, $X(s,t)$ is a random variable

4. **What is discrete Random sequence? Give an example.**
   If both state space S and parameter set T are discrete, then the random process is called a discrete random sequence. If X_n represents the outcome of the nth toss of a fair die, then $\{X_n,\ n \geq 1\}$ is discrete random sequence.

5. **Define discrete random process. Give example**
   If T is continuous and S is discrete, the random process is called a discrete   random process.

   Example:

   If X(t) represents the number of telephone calls received in the interval (o, t), then $\{X(t)\}$ is a discrete random process.

6. **What is a continuous random process. Give example.**
   If both S and T are continuous, the random process is called a continuous random process.

Example: If X(t) represents the maximum temperature at a place in the interval (o, t), $\{X(t)\}$ is a continuous random process.

7. What do you mean by the Mean and Variance of a random process?

If $X(t)$ is a representative members function of the random process $\{X(t)\}$, $\mathrm{E}[X(t)]$ and $Var[X(t)]$ are called the Mean and Variance of the process

8. When are two random processes said to be orthogonal?

Two random processes $\{X(t)\}$ $\{Y(t)\}$ are said to be orthogonal. If

$$\mathrm{E}[X(t_1)Y(t_2)] = 0$$

9. Define the Auto correlation and Auto ovariance of Random process $\{X(t)\}$

Autocorrelation Function

$$R_{XX}(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)]$$

Auto covariance

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mathrm{E}[X(t_1)]\mathrm{E}[X(t_2)]$$

10. Define the Cross Correlation of two random process.

$$R_{XY}(t_1, t_2) = \mathrm{E}[X(t_1)Y(t_2)]$$

11. Define Cross Covariance of two random processes

$$R_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mathrm{E}[X(t_1)]\mathrm{E}[Y(t_2)]$$

12. Define a Stationary Process

If certain probability distribution or averages do not depend on time t, then the random process $\{X(t)\}$ is called stationary process.

13. Define Strict Sense Stationary Process.

A random process $\{X(t)\}$ is a SSS process. If the joint distribution of $X(t_1), X(t_2), ...X(t_n)$ is the same as that of $X(t_1 + h), X(t_2 + h), ...X(t_n + h)$ for all $t_1, t_2, ...t_n$ and h > 0 and for all n ≥ 1

Example: Bernoulli's Process.

14. Define Second order Stationary process

A random process $\{X(t)\}$ is said to be second order SSS, if

$$f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1 + h, t_2 + h)$$ where $f(x_1, x_2, t_1, t_2)$ is the joint p.d.f $\{X(t_1), X(t_2)\}$

15. Define Wide Sense Stationary Process

A random process $\{X(t)\}$ is called wide sense stationary. If $E[X(t)]$ is a constant and $E[X(t)X(t+\tau)] = R_{XX}(z)$

(i.e) A.C.F. is a function of Z only.

16. Define evolutionary process and give example.

A random process $\{X(t)\}$ that is not stationary in any sense is called an evolutionary process.

Example: Poisson process.

17. When are the processes $\{X(t)\}$ and $\{Y(t)\}$ said to be jointly stationary in the Wide Sense?

Two random process $\{X(t)\}$ and $\{Y(t)\}$ said to be jointly stationary in the Wide sense if each process is individually a WSS process and (t_1, t_2) is a function of (t_1, t_2) only.

18. State any four properties of Auto Correlation Function

(1) $R_{XX}(-Z) = R_{XX}(Z)$

(2) $|R(Z)| \leq R(0)$

(3) $R(Z)$ is continuous for all Z

(4) If $R(Z)$ is A.C.F. of a stationary R.P. $\{X(t)\}$ with no periodic components, then

$$\mu_X{}^2 = \lim_{z \to \infty} R(Z)$$

19. Give that the Auto correlation function for a stationary ergodic process with no periodic components is $R(Z) = 25 + \dfrac{4}{1+6Z^2}$. Find the Mean and variance of the Process $\{X(t)\}$

Soln:

$$\mu_X{}^2 = \lim_{z \to \infty} R(Z) = 25$$
$$= 5$$

$$E[X^2(t)] = R_{XX}(0) = 25 + 4$$
$$= 29$$

$$Var[X(t)] = E[X^2(t)] - E^2[X(t)]$$
$$= 29 - 25$$
$$= 4$$

20. Find the Mean and Variance of the stationary process $\{X(t)\}$ whose

A.C.F. $R(Z) = \dfrac{25Z^2 + 36}{6.25Z^2 + 4}$

Soln:

$$\mu_X{}^2 = \lim_{z \to \infty} R(Z)$$

$$= \lim_{T \to \infty} \frac{25 + 36/Z^2}{6.25 + 4/Z^2}$$

$$= 4$$

$$\mu_X = 2$$

$$E[X^2(t)] = R_{XX}(0)$$
$$= 9$$

$$V[X(t)] = 9.4$$
$$= 5$$

21. A stationary random process [X(t)] with mean 4 has ACF R(Z) = 16 + 9 $e^{-|z|}$. Find the standard deviation of the process

$$E[X^2(t)] = R_{XX}(o)$$
$$= 16 + 9 = 25$$
$$V[X(t)] = 25 - 16 = 9$$

Standard deviation = 3

22. Define the Cross- Correlation function and state any two of its properties.
    Soln:

$$R_{XY}(t_1,t_2) = E[X(t_1)X(t_2)]$$

$$(1) R_{XY}(Z) = R_{XX}(-Z)$$

$$(2) |R_{XY}(Z)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$\leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

23. When is a random process said to be ergodic? Give example?

A random process $\{X(t)\}$ is ergodic if its ensembled averages equal to appropriate time averages.

Example: $X(t) = A\cos(\omega t + \theta)$ $\theta =$ uniformly distributed in $(0, 2\pi)$ is Mean ergodic.

24. Distinguish between stationary and ergodicity.

Stationary of a random process is the property of the process by which certain probability distributions or averages do not depend on time t.

Ergodocity of a random process is the property by which almost every member of the process exhibits the same statistical behavior as the whole process.

25. Check for the stationary of the random process $X(t) = A\cos(\omega t + \theta)$ if A and ω_0 are constants and θ is a uniformly distributed RV in (0, 2π)

$$E[X(t)] = 0 \text{ and } R_{XX}(t,t+z) = \frac{A^2}{2}\cos \omega_0 z$$

$\Rightarrow [X(t)]$ is wide sense stationary.

26. Give that R.P. $\{X(t)\} = A\cos \omega_0 t + B\sin \omega_0 t$ where ω_0 is constant and A and B are uncorrelated zero mean random variables having different densities but the same variance. Is $\{X(t)\}$ wide sense stationary?

$$E[X(t)] = 0$$

$$E[X^2(t)] = \sigma^2$$

$$and \ R_{XX}(t_1,t_2) = \sigma^2 \cos \omega_0(t_1 - t_2)$$

$X(t)$ is Wide sense stationary.

27. Define Markov Process.

If the future behaviour of a Process depends only the present state but not on the past, the process is a Markov process.

28. Define Markov chain

A discrete parameter Markov process is called a Markov chain
(or)If,forall

$$P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ...X_0 = a_0] = P[X_n = a_n / X_{n-1} = a_{n-1}]$$

then the process $\{X_n\}$, n= 0, 1, 2, ... Is called Markov Chain

$(a_1, a_2, ...a_n, ...)$ called the states of the Markov Chain.

29. Define A Markov Process.

If for $t_1 < t_2 < ...... < t_n < t,$

$$P[X(t) \le x / X(t_1) = x_1, X(t_2) = x_2, ...X(t_n) = x_n] = P[X(t) \le x / X(t_n) = x_n]$$

Then the process $\{X(t)\}$ is called Markov Process.

30. What is a Stochastic matrix? When is it said to be regular.

If $P_{ij} \ge 0$ and $\sum P_{ij} = 1$ for all I, then the t.p.m. $P = (P_{ij})$ of a

Markovchain is a stochastic matrix P. A stochastic matrix P is said to be regular matrix, if all the entries of P^m (for some +ve integer m) are positive.

31. Define irreducible Markov chain. Also state Chapman – Kolmogrov Theorem.

If $P_{ij}^{(n)} > 0$ for some n and for all i and j, then every state can be

reached from every other state. When this condition is satisfied, the Markov Chain is said to be irreducible

Theorem:

If P is the t.p.m. of a homogeneous Markov chain, then n step t.p.m. $P^{(n)}$ is

equal to p^n .

32. What is a Markov chain? When can you say that a Markov chain is homogeneous?

A discrete state stochastic process $\{X(t) / t \in T\}$ is a Markov Chain.

If

$$[X(t) \le x / X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, ...X(t_0) = x_0] = P[X(t) \le x / X(t_n) = x_n]$$

If it has the property of invariance with respect to the time origin.

(i.e) $P[X(t) \leq x / X(t_n) = x_n] = P[X(t - t_n) \leq x / X(0) = x_n]$,
then the Markov Chain is homogeneous

.

33. Consider the random process $\{X(t)\} = \cos(\omega_0 t + \theta)$ **where θ is uniformly distributed in the interval (-π, π). Check whether X(t) is stationary or not?**

$$E[X(t)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) d\theta$$

$$= \frac{1}{2\pi} [\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)]$$

$$= \frac{1}{2\pi} [-\sin \omega_0 t + \sin \omega_0 t]$$

$$= 0$$

$$E[X^2(t)] = \frac{1}{4\pi} \left[\frac{\theta - 2\sin(\omega_0 t + \theta)}{2} \right]$$

$$= \frac{1}{2}$$

The process is stationary as the first and second moments are independent of time.

34. The one- step t.p.m. of a Markov Chain with states 0 and 1 is given as

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **Draw the transition diagram. Is it irreducible Markov chain?**

Yes, it is irreducible, because each state can be reached from any other state.

35. Three boys A, B, C are throwing a ball to each other. A always throw the ball to B and B always throws the ball to C but C is just as likely to throw the ball B as to A. Find the transition Matrix and show that the process is Markovian.

TPM of $\{X(t)\}$ is given by P

$$X_n$$

$$P = \begin{matrix} \\ X_{n-1} \\ \\ \end{matrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

Since X_n depends only on X_{n-1} but not on states of $X_{n-2}, X_{n-3}, \ldots\ldots \{X_n\}$ is a Markov process.

36. What do you mean by an absorbing Markov chain? Give an example.

A state i of a Markov chain is said to be an absorbing state if $P_{ij} = 1$.(i.e.) if it is impossible to leave it. A markov chain is said to be absorbing if it has atleast one absorbing state.

The TPM of an absorbing Markov Chain is

$$tpm = \begin{matrix} & 1 & 2 & 3 & 4 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

37. Prove that the matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ **is the t.p.m. of an irreducible Markov chain.**

$$P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$P_{11}^{(3)} > 0, P_{13}^{(2)} > 0, P_{21}^{(2)} > 0, P_{22}^{(2)} > 0, P_{33}^{(2)} > 0$ and for all other $P_{ij}^{(1)} > 0$, the chain is irreducible.

38. State the postulates of a poisson process.

Let X(t) = number of times an event A say occurred upto time 't' so that the sequence $\{X(t)\}$, $t \geq 0$ forms a Poisson process with parameter λ.

(i)   Events occurring in non- overlapping intervals are independent of each other

(ii)   P[X (t) =1 for t in (x , x + h) = λh + 0(h)]

(iii)   P[X(t) = 0 for t in (x , x + h) = 1- λh + 0(h)]

(iv)   P[X(t) = 2 or more for t in (x , x + h)] = 0]

(v)

39. State any two properties of Poisson Process.

(i)   The Poisson process is a Markov Process.

(ii)   Sum of two independent Poisson processes is a Poisson process.

(iii)   The difference of two independent Poisson processes is not a Poisson process.

40. What will be the super position of n independent Poisson processes with respective averages rates $\lambda_1, \lambda_2, ...\lambda_n$?

The super position of n independent Poisson processes with averages rates $\lambda_1, \lambda_2, ...\lambda_n$ is another Poisson process with averages rate

$$\lambda_1 + \lambda_2 + ... + \lambda_n$$

41. If the customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute. Find the probability that the interval between two consecutive arrivals is more than one minute?

The interval T between 2 consecutive arrivals follows an exponential distribution with parameter λ = 2, $P[T >1] = \int\limits_1^\infty 2e^{-2t} = e^{-2} = 0.135$

42. Let X(t) be a Poisson process with rate λ. Find correlation function of X (t).

43.   For Z > 0,

$$E[X(t).X(t+z)] = E\{X(t)[X(t+z) - X(t) + X(t)]\}$$
$$= E[X(t)]E[X(t+z) - X(t)] + E[X^2(t)]$$
$$= \lambda t(\lambda z) + \lambda^2 t^2 + \lambda t$$
$$= \lambda^2 tz + \lambda^2 t^2 + \lambda t$$

44. Show that the Poisson process is not Covariance stationary?

$$P[X(t)] = \frac{e^{-\lambda t}(\lambda t)^r}{r!}, r = 0,1,2,....$$

E[X(t)] = λt $\neq$ a constant.   The process is not covariance stationary.

45. A bank receives on the average λ = 6 bad checks per day, what are the probabilities that it will receive (i) 4 bad checks on any given day, (ii) 10 bad checks over any two consecutive days?

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \frac{e^{-6t}(6t)^n}{n!}, n = 0,1,2,....$$

$$(i)P[X(1) = 4] = \frac{e^{-6}(6)^4}{4!} = 0.1338$$

$$(ii)P[X(2) = 10] = \frac{e^{-12}(12)^{10}}{10!} = 0.1048$$

46. **Customers arrive a large store randomly at an average rate of 240 per hour. Shat is the probability that during a two- minute interval no one will arrive?**

$$P[X(t) = n] = \frac{e^{-4t}(4t)^n}{n!}, n = 0,1,2,...$$

$$\text{since } \lambda = \frac{240}{60} = 4$$

$$P[X(2) = 0] = e^{-8} = 0.0003$$

47. **The number of arrivals at the Reginal computer center at express service counter between 12 noon and 3 pm has a poisson distribution with a Mean of 1.2 perminute. Find the probability of no arrivals during a given 1- minute interval.**

$$P[X(t) = n] = \frac{e^{(1.2)t}\{1.2t\}^n}{n!}, n = 0,1,2,...$$

$$P[X(1) = 0] = e^{-1.2} = 0.3012$$

48. **Define Gaussian or normal process (or) when is a random process is said to be normal?**

A real valued R.P. $\{X(t)\}$ is called a Gaussian process or normal process if the random variables $X(t_1), X(t_2),.........X(t_n)$ are jointly normal for any n and for any set $t_i's$

49. **State the properties of Gaussian process.**
    (i)   If a Gaussian process is wide sense stationary, it is also strict sense stationary
    (ii)   If the member functions of a Gaussian process are uncorrelated, the they are independent.
    (iii)   If the input $\{X(t)\}$ of a linear system is a Gaussian process, the output will also be a Gaussian process.

50. **Name a few random processes that are defined interms of stationary Gaussian random process.**
    (i)   Square law detector process
    (ii)   Full- Wave linear detector process
    (iii)   Half- Wave linear detector process
    (iv)   Hard limiter process.

51. **If $\{X(t)\}$ is a Gaussian process with $\mu(t) = 10$ and**

$$C(t_1,t_2) = 16e^{-|t_1-t_2|} \text{ Find } P\{X(10) \le 8\}$$

$\mu[X(10)] = 10$ and Var[X(10)]=C(10,10)=16

$$P[X(10) \le 8] = P\left[\frac{X(10)-10}{4} \le -0.5\right]$$

$$= P[z \le -0.5]$$
$$= 0.5 - P[z \le 0.5]$$
$$= 0.5 - 0.1915$$
$$= 0.3085$$

52. **Given a normal process $\{X(t)\}$ with zero mean and $R_{XX}(z) = 4e^{-2|z|}$. Find Var[X(t)].**

$$Var[X(t)] = R_{XX}(z) - E^2[X(t)]$$
$$= 4e^{-2|z|}$$

53. **If X(t) is a normal process with $C(t_1,t_2) = 4e^{-0.5|z|}$, what is the variance of X(5).**

$$Var[X(t)] = C(5,5) = 4e^{-0.5(0)}$$
$$= 4, \qquad \text{since z=} t_1 - t_2$$

54. **If $\{X(t)\}$ is a normal process with $\mu(t) = 3$ and $C(t_1 - t_2) = 4e^{-0.2|t_1-t_2|}$ find the variance of X(8) – X(5)**

$$Var[X(8) - X(5)] = var[X(8) + X(5)] - 2\,cov[X(8), X(5)]$$
$$= 4 + 4 - 2 \text{ X4 X e}^{-0.6}$$
$$= 8[1 - e^{-0.6}]$$
$$= 3.6095$$

PART B

1. The process { X(t) } whose proabability distribution is given by

$$P\,[\,X(t) = n] = \frac{(at)^{n-1}}{(1+at)^{n+1}}\,, n = 1, 2...$$

$$= \frac{(at)}{1+at}, n = 0 \qquad \text{Show that it is not stationary}$$

2. A raining process is considered as a two state Markov chain. If it rains, it is considered to be in state 0 and if it does not rain, the chain is in state 1. the transitioin probability of the markov chain is defined as $P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$. Find the probability of the Markov chain is defined as today assuming that it is raining today. Find also the unconditional probability that it will rain after three days with the initial probabilities of state 0 and state 1 as 0.4 and 0.6 respectively.

3. Let X(t) be a Poisson process with arrival rate λ. Find E $\{(\,X\,(t) - X\,(s)^2\,\}$ for t > s.

4. Let { X_n ; n = 1,2..} be a Markov chain on the space S = { 1,2,3} with on step transition probability matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$ (1) Sketch transition diagram (2) Is the chain irreducible? Explain. (3) Is the chain Ergodic? Explain

5. Consider a random process X(t) defined by X(t) = U cost + (V+1) sint, where U and V are independent random variables for which E (U) = E(V) = 0 ; E (U ^{2}) = E (V^2) = 1   (1) Find the auto covariance function of X (t) (2) IS X (t) wide sense stationary? Explain your answer.

6. Discuss the pure birth process and hence obtain its probabilities, mean and variance.

7. At the receiver of an AM radio, the received signal contains a cosine carrier signal at the carrier frequency ω with a random phase θ that is uniform distributed over ($0,2\pi$). The received carrier signal is X (t) = A cos(ωt + θ). Show that the process is second order stationary

8. Assume that a computer system is in any one of the three states busy, idle and under repair respectively denoted by 0,1,2. observing its state at 2 pm each day, we get the transition probability matrix as $P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{pmatrix}$. Find out the 3rd step transition probability matrix. Determine the limiting probabilities.

9. Given a random variable Ω with density f (ω) and another random variable φ uniformly distributed in (-π, π) and independent of Ω and X (t) = a cos (Ωt + φ), Prove that { X (t)} is a WSS Process.

10. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work iff a 6 appeared. Find (1) the probability that he takes a train on the 3rd day. (2) the probability that he drives to work in the long run.

11. Show that the process X (t) = A cosλt + B sin λt (where A and B are random variables) is wide sense stationary, if (1) E (A) = E(B) = 0 (2) E(A^2) = E (B^2) and E(AB) = 0

12. Find probability distribution of Poisson process.

13. Prove that sum of two Poisson processes is again a Poisson process.

14. Write classifications of random processes with example